Animating orientation

CS 448D: Character Animation
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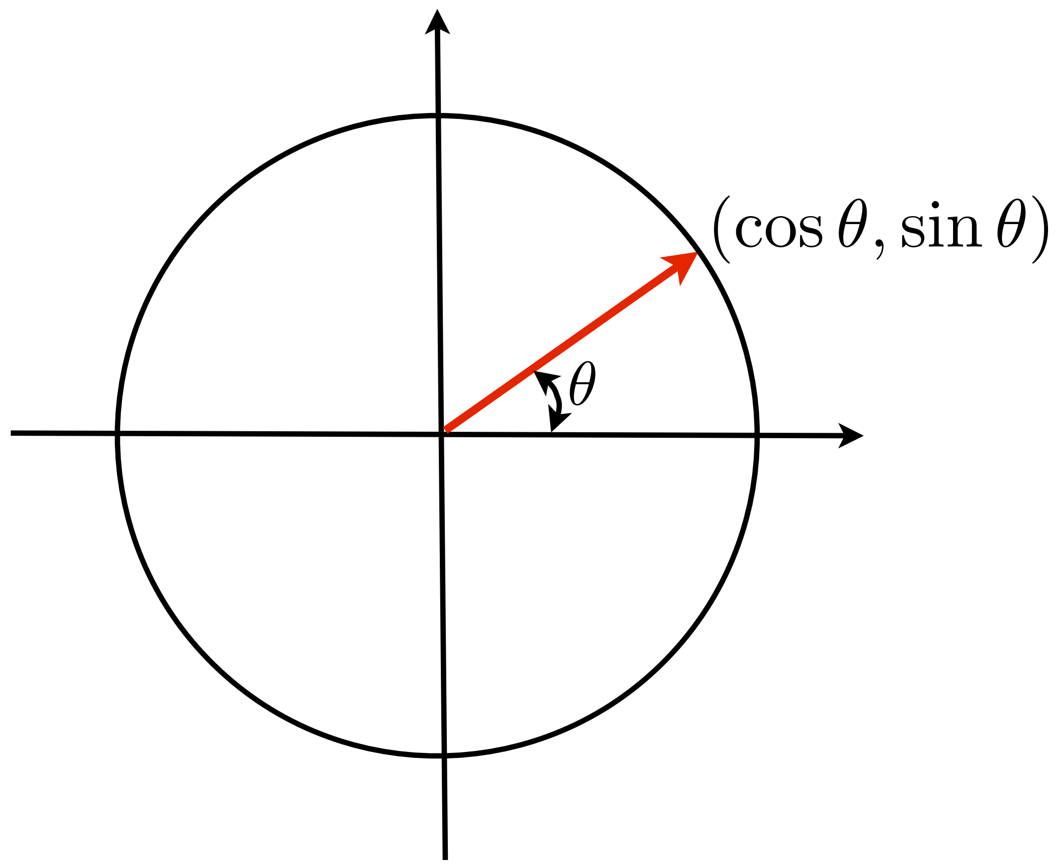
Stanford University

Orientation in the plane



$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Refresher: Homogenous coordinates

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix}$$

$$T_{t_x,t_y,t_z}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix}$$

$$S_{s_x,s_y,s_z}(\mathbf{v}) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix}$$

Orientation in 3D

$$R^x_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R^y_\beta = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R^z_\gamma = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Orientation in 3D

Any orientation in 3D can be represented as a combination of three angles, specifying three consecutive rotations around axes.

$$R_{\alpha,\beta,\gamma} = R^z_\gamma R^y_\beta R^x_\alpha$$

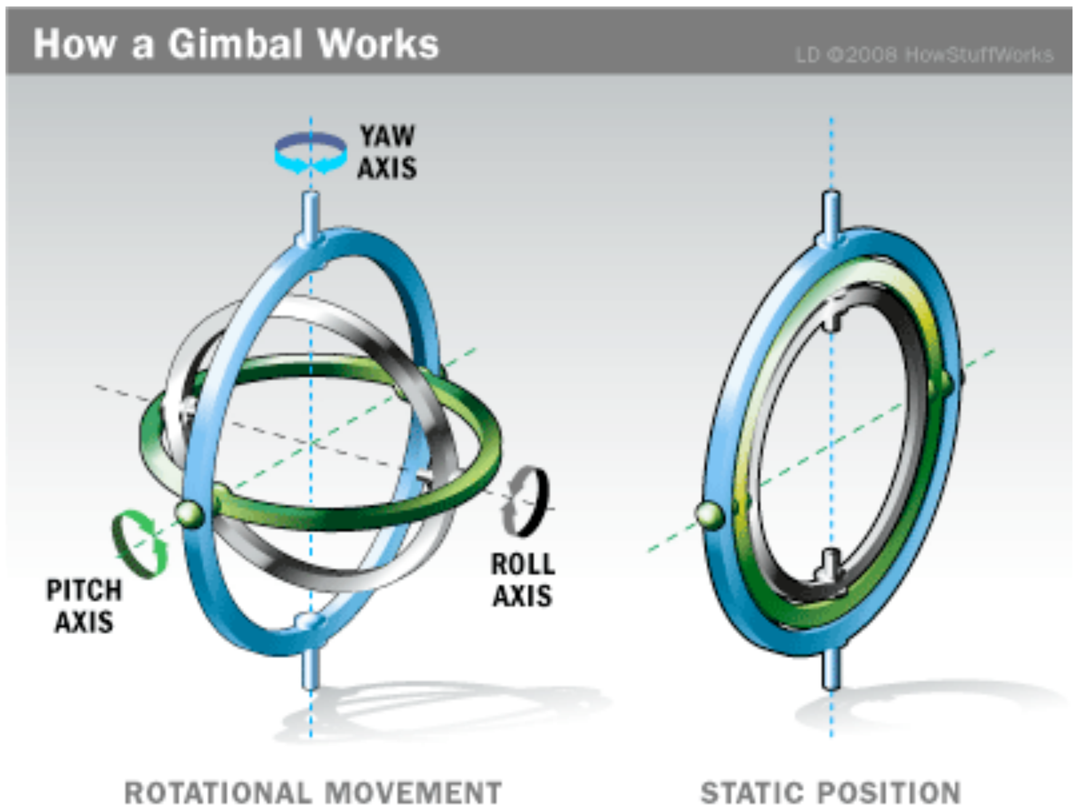
or $R_{\alpha,\beta,\gamma} = R^x_\alpha R^y_\beta R^z_\gamma$?

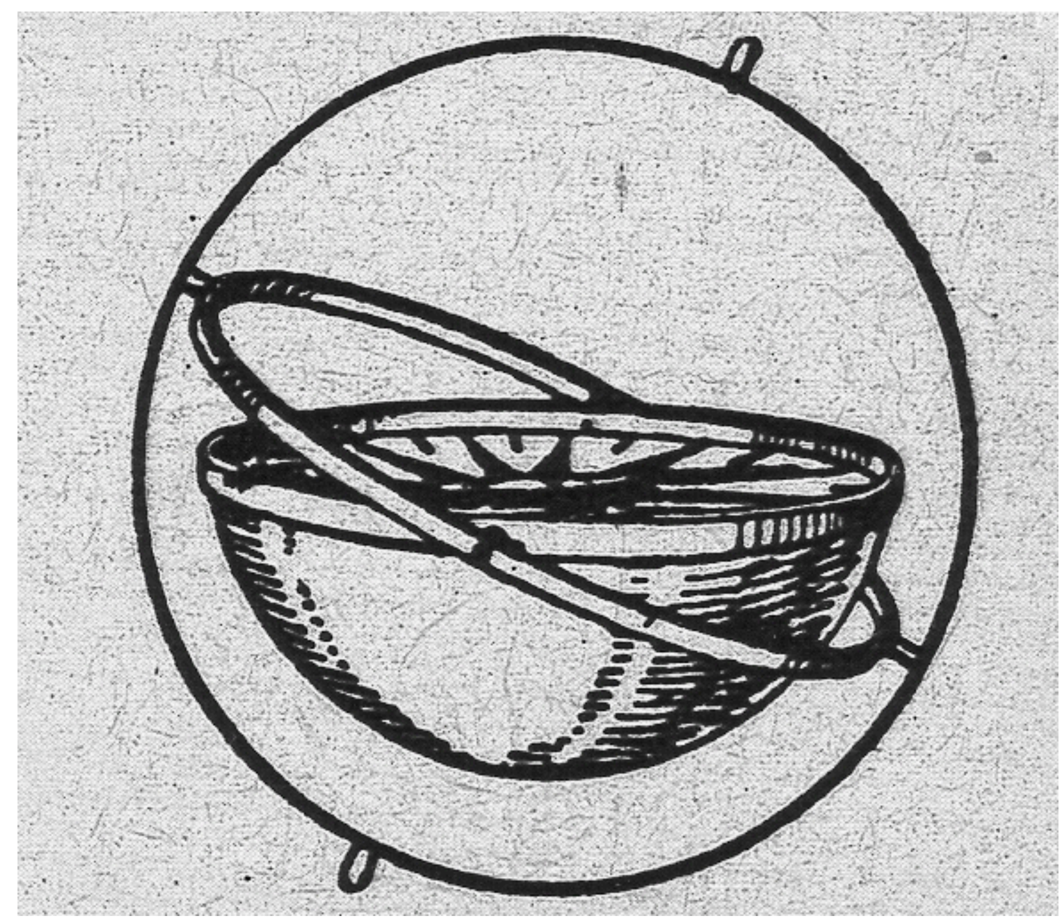
or $R_{\alpha,\beta,\gamma} = R^z_\alpha R^x_\beta R^z_\gamma$?

These are fixed-angle or Euler angle representations.
(Equivalent up to changing the order.)

Interpolation and gimbal lock

Gimbal


How a Gimbal Works
LD ©2008 HowStuffWorks
YAW AXIS
PITCH AXIS
ROLL AXIS
ROTATIONAL MOVEMENT
STATIC POSITION



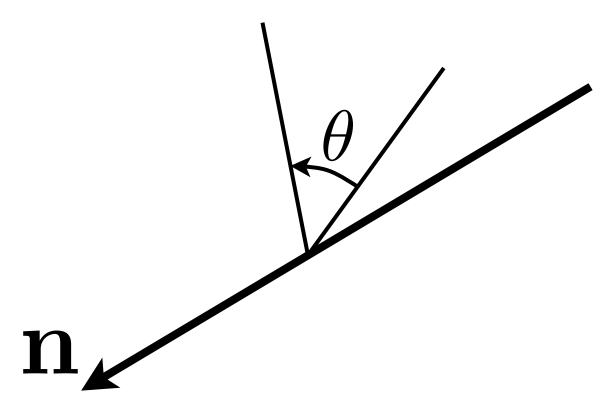
Interpolation and gimbal lock

interactive demonstration

Euler’s theorem

(well, one of them)

Any orientation can be specified by a rotation of angle θ about an axis $\mathbf{n}$



Quaternions

- An elegant representation of rotation in terms of axis and angle
- Interpolates smoothly
- Easy to compose

Quaternions

Higher-dimensional complex numbers

$$q = s + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$$

$$q = (s, x, y, z)$$

$$q = (s, \mathbf{v})$$

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$$

ij = k	ji = -k
jk = i	kj = -i
ki = j	ik = -j

Quaternion arithmetic

$$
\begin{aligned}
q + q' &= (s, \mathbf{v}) + (s', \mathbf{v}') \\
&= \big(s + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}\big) + \big(s' + x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}\big) \\
&= (s + s') + (x + x')\mathrm{i} + (y + y')\mathrm{j} + (z + z')\mathrm{k} \\
&= (s + s', \mathbf{v} + \mathbf{v}')
\end{aligned}
$$

$$
\begin{aligned}
qq' &= (s, \mathbf{v})(s', \mathbf{v}') \\
&= \big(s + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}\big)\big(s' + x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}\big) \\
&= ss' - \big(xx' + yy' + zz'\big) + s\big(x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}\big) + s'\big(x\mathrm{i} + y\mathrm{j} + z\mathrm{k}\big) \\
&+ (yz' - zy')\mathrm{i} + (zx' - xz')\mathrm{j} + (xy' - yx')\mathrm{k}
\end{aligned}
$$

Quaternion arithmetic

$$
\begin{aligned}
q + q' &= (s, \mathbf{v}) + (s', \mathbf{v}') \\
&= \big(s + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}\big) + \big(s' + x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}\big) \\
&= (s + s') + (x + x')\mathrm{i} + (y + y')\mathrm{j} + (z + z')\mathrm{k} \\
&= (s + s', \mathbf{v} + \mathbf{v}')
\end{aligned}
$$

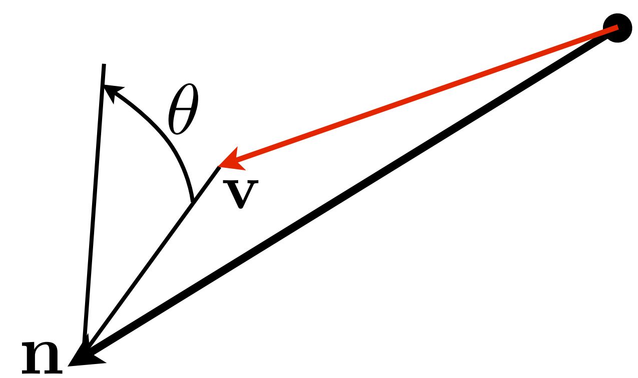
$$
\begin{aligned}
qq' &= (s, \mathbf{v})(s', \mathbf{v}') \\
&= \big(s + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}\big)\big(s' + x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}\big) \\
&= ss' - \big(xx' + yy' + zz'\big) \\
&\quad + s\big(x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}\big) + s'\big(x\mathrm{i} + y\mathrm{j} + z\mathrm{k}\big) \\
&\quad + (yz' - zy')\mathrm{i} + (zx' - xz')\mathrm{j} + (xy' - yx')\mathrm{k} \\
&= \big(ss' - \mathbf{v}\mathbf{v}',\ \mathbf{v} \times \mathbf{v}' + s\mathbf{v}' + s'\mathbf{v}\big)
\end{aligned}
$$

Quaternion multiplicative inverse

$$q^{-1} = \frac{(s, -\mathbf{v})}{\|q\|^2}$$

$$qq^{-1} = \frac{(s, \mathbf{v})(s, -\mathbf{v})}{\|q\|^2} = \frac{s^2 + \|\mathbf{v}^2\|}{\|q\|^2} = 1$$

Representing rotation with quaternions



$$q = \left(\cos\tfrac{\theta}{2}, \mathbf{n}\sin\tfrac{\theta}{2}\right)$$

$$\|\mathbf{n}\| = 1$$

$$p = (0, \mathbf{v})$$

$$R_{\theta,\mathbf{n}}(\mathbf{v}) = qpq^{-1}$$

$$q^{-1} = \left(\cos\tfrac{\theta}{2}, -\mathbf{n}\sin\tfrac{\theta}{2}\right)$$

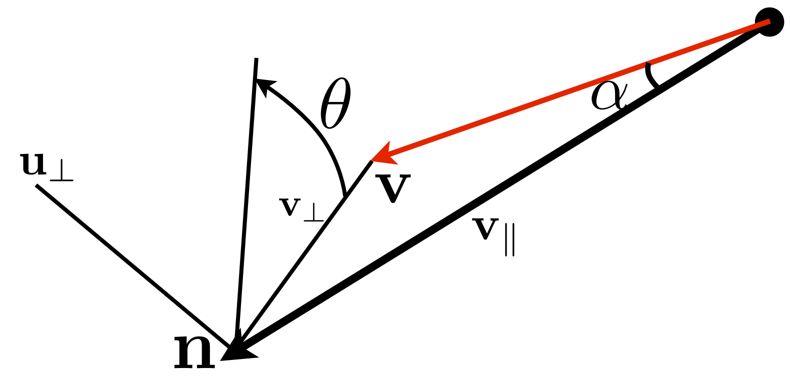
Representing rotation with quaternions

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\cos 2\alpha &= 1 - 2\sin^2\alpha
\end{aligned}$$

$$\begin{aligned}
qpq^{-1} &= \left(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}\mathbf{n}\right)(0,\mathbf{v})\left(\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}\mathbf{n}\right) \\
&= \left(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}\mathbf{n}\right)\left(\sin\tfrac{\theta}{2}\mathbf{vn}, -\sin\tfrac{\theta}{2}(\mathbf{v}\times\mathbf{n}) + \cos\tfrac{\theta}{2}\mathbf{v}\right) \\
&= \left(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}\mathbf{n}\right)\left(\sin\tfrac{\theta}{2}\mathbf{vn}, \sin\tfrac{\theta}{2}(\mathbf{n}\times\mathbf{v}) + \cos\tfrac{\theta}{2}\mathbf{v}\right) \\
&= \Big(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\mathbf{vn} - \sin^2\tfrac{\theta}{2}\mathbf{n}(\mathbf{n}\times\mathbf{v}) - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\mathbf{nv}, \\
&\qquad \sin^2\tfrac{\theta}{2}\mathbf{n}\times(\mathbf{n}\times\mathbf{v}) + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{n}\times\mathbf{v}) + \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{n}\times\mathbf{v}) + \cos^2\tfrac{\theta}{2}\mathbf{v} + \sin^2\tfrac{\theta}{2}\mathbf{n}(\mathbf{vn})\Big) \\
&= \Big(0, \sin^2\tfrac{\theta}{2}\big(\mathbf{n}(\mathbf{nv}) - \mathbf{v}(\mathbf{nn})\big) + 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{n}\times\mathbf{v}) + \cos^2\tfrac{\theta}{2}\mathbf{v} + \sin^2\tfrac{\theta}{2}\mathbf{n}(\mathbf{vn})\Big) \\
&= \Big(0, 2\sin^2\tfrac{\theta}{2}\mathbf{n}(\mathbf{vn}) + 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}(\mathbf{n}\times\mathbf{v}) + \cos^2\tfrac{\theta}{2}\mathbf{v} - \sin^2\tfrac{\theta}{2}\mathbf{v}\Big) \\
&= \Big(0, (1-\cos\theta)\mathbf{n}(\mathbf{vn}) + \sin\theta(\mathbf{n}\times\mathbf{v}) + \cos\theta\mathbf{v}\Big)
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}\times\mathbf{u} &= -\mathbf{u}\times\mathbf{v} \\
\mathbf{v}\times(\mathbf{u}\times\mathbf{w}) &= \mathbf{u}(\mathbf{vw}) - \mathbf{w}(\mathbf{vu})
\end{aligned}$$

Representing rotation with quaternions



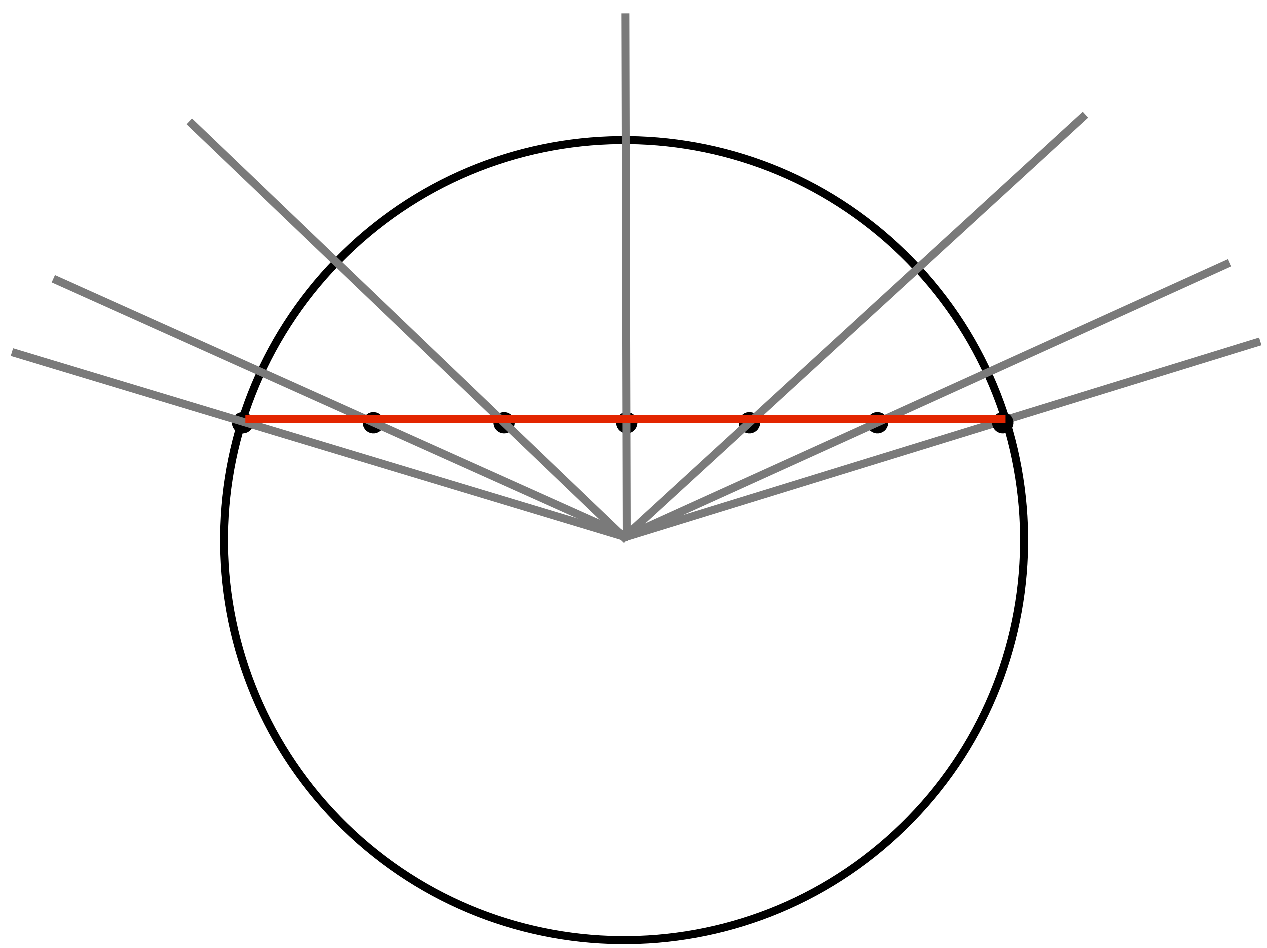
Now what does rotation of v by θ about n actually do?

$$\begin{aligned}
R_{\theta,\mathbf{n}}(\mathbf{v}) &= R_{\theta,\mathbf{n}}(\mathbf{v}_{||}) + R_{\theta,\mathbf{n}}(\mathbf{v}_\perp) \\
&= \mathbf{v}_{||} + (\cos\theta)\mathbf{v}_\perp + (\sin\theta)\mathbf{u}_\perp \\
&= \mathbf{n}(\mathbf{vn}) + (\cos\theta)\big(\mathbf{v} - \mathbf{n}(\mathbf{vn})\big) + (\sin\theta)\mathbf{v}\times\mathbf{n} \\
&= (1-\cos\theta)\mathbf{n}(\mathbf{vn}) + (\cos\theta)\mathbf{v} + (\sin\theta)\mathbf{v}\times\mathbf{n} \\
&= qpq^{-1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}\times\mathbf{n} &= \|\mathbf{v}\|\|\mathbf{n}\|\sin\alpha\frac{\mathbf{u}_\perp}{\|\mathbf{u}_\perp\|} \\
\mathbf{vn} &= \|\mathbf{v}\|\|\mathbf{n}\|\cos\alpha
\end{aligned}$$

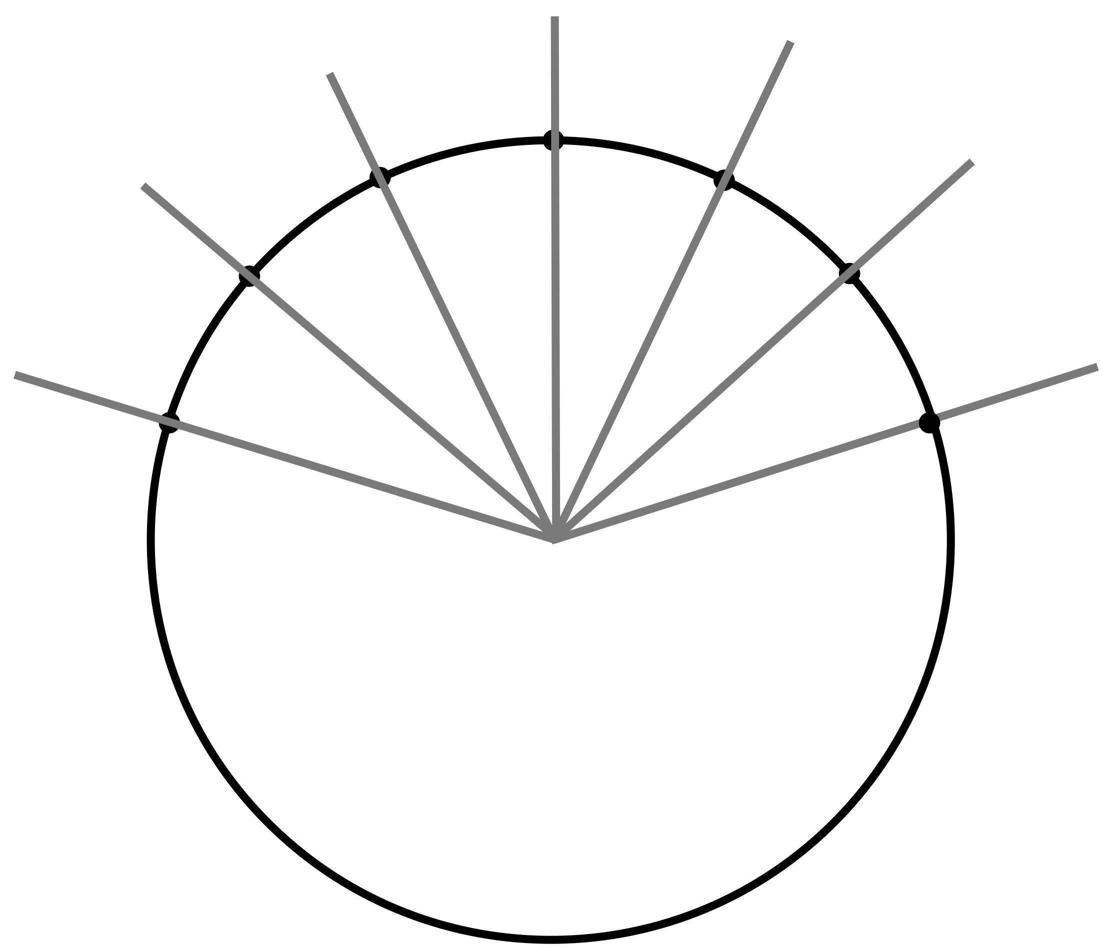
Interpolating quaternions

- Quaternions that represent rotation as described so far lie on the unit sphere in the four-dimensional quaternion space.
- Any quaternion q represents rotation, the same as $q/||q||$.
- We can linearly interpolate between two quaternions by treating them as generic four-dimensional vectors, but the interpolation would speed up in the middle.



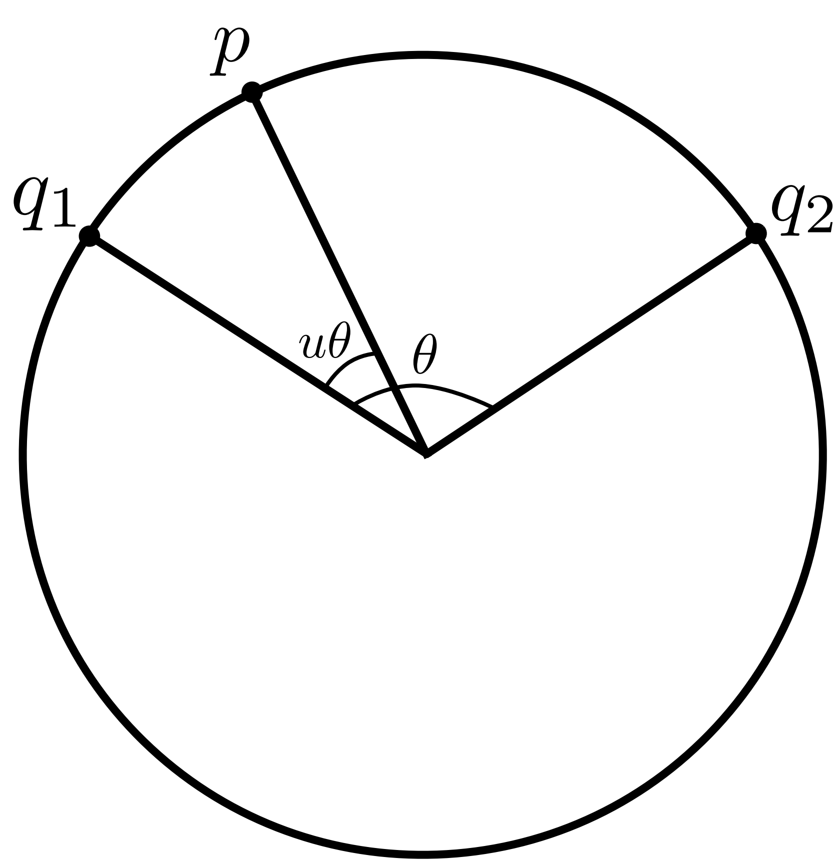
Interpolating quaternions

- Instead we interpolate on the unit sphere.
- This results in smooth uniform motion.



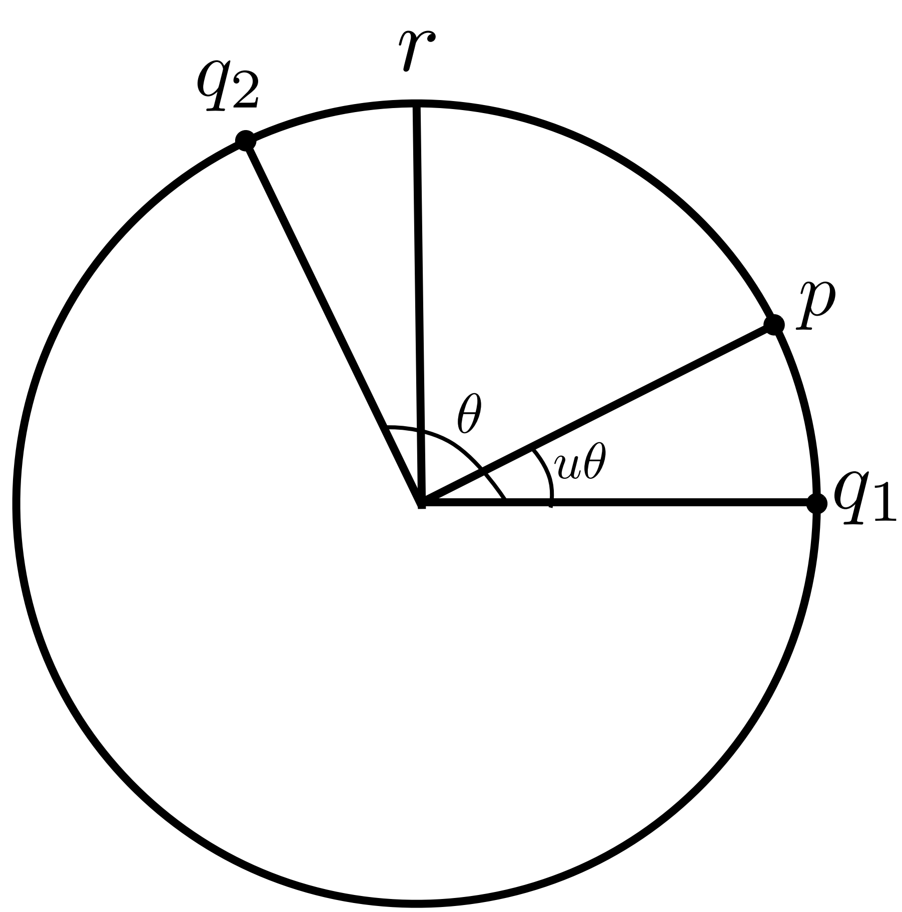
Spherical Linear Interpolation (slerp)

$$p = \mathrm{slerp}(q_1, q_2, u) =?$$



Spherical Linear Interpolation (slerp)

$$
\begin{aligned}
p = \mathrm{slerp}(q_1, q_2, u) &= \cos(u\theta) q_1 + \sin(u\theta) r \\
&= \cos(u\theta) q_1 + \sin(u\theta) \frac{q_2 - \cos(\theta) q_1}{\sin(\theta)} \\
&= \frac{\sin(\theta)\cos(u\theta) - \cos(\theta)\sin(u\theta)}{\sin(\theta)} q_1 + \frac{\sin(u\theta)}{\sin(\theta)} q_2 \\
&= \frac{\sin((1-u)\theta)}{\sin(\theta)} q_1 + \frac{\sin(u\theta)}{\sin(\theta)} q_2
\end{aligned}
$$



$$
\begin{aligned}
q_2 &= \cos(\theta) q_1 + \sin(\theta) r \\
r &= \frac{q_2 - \cos(\theta) q_1}{\sin(\theta)}
\end{aligned}
$$

Interpolating through the smaller angle angle

q and -q represent the same rotation.

$$
\begin{aligned}
q &= \left(\cos\tfrac{\theta}{2}, \mathbf{n}\sin\tfrac{\theta}{2}\right) \\
-q &= \left(-\cos\tfrac{\theta}{2}, -\mathbf{n}\sin\tfrac{\theta}{2}\right) \\
&= \left(\cos\left(\pi + \tfrac{\theta}{2}\right), -\mathbf{n}\sin\left(\pi + \tfrac{\theta}{2}\right)\right) \\
&= \left(\cos\tfrac{2\pi+\theta}{2}, \mathbf{n}\sin\tfrac{2\pi+\theta}{2}\right)
\end{aligned}
$$

To interpolate from p to q through the smaller angle, compute the distances ||p-q|| and ||p+q|| and choose the smaller one.

Higher orders of continuity

Bezier curves on the unit sphere of quaternions.

See Shoemake, "Animating Rotation with Quaternion Curves," SIGGRAPH 1985, for details.

Conversions

Animators still specify orientation keys in Euler angles.

Euler angles provide a visually intuitive and familiar interface.

They are fine for specifying individual keys, just not interpolation. Interpolation is done in quaternions.

We need to regularly convert between rotation matrices and quaternions.

Conversions: Quaternions to matrices

$$q = (s, x, y, z)$$

$$A = \begin{pmatrix} 1-2(y^2+z^2) & 2xy-2sz & 2sy+2xz & 0 \\ 2xy+2sz & 1-2(x^2+z^2) & -2sx+2yz & 0 \\ -2sy+2xz & 2sx+2yz & 1-2(x^2+y^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

See Shoemake, "Animating Rotation with Quaternion Curves," SIGGRAPH 1985, for details.

Conversions: Matrices to quaternions

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$q = (s, x, y, z)$$

$$\begin{aligned} s &= \pm\frac{1}{2}\sqrt{A_{00} + A_{11} + A_{22} + A_{33}} \\ x &= \frac{A_{21} - A_{12}}{4s} \\ y &= \frac{A_{02} - A_{20}}{4s} \\ z &= \frac{A_{10} - A_{01}}{4s} \end{aligned}$$

follows from previous slide by simple arithmetic, remembering that ||q||=1.